Talk Notes

Thursday, June 11, 2020 3:40 PM

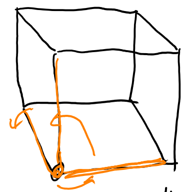
Groups as symmetries of objects - Following OH 1 from the text Office Hours with a geometric group theorist + bits of later OH.

Today
- General discussion of groups → symmetries
- Infinite groups
- Group homomorphisms
- Group presentations
- Examples: Coxeter, Lamplighter, Braid

Groups Study symmetries - better understand the object - interaction spaces - groups heart of geometric group theory.

Example The cube

1



rigid transformations pick the cube and rotate it in your hands

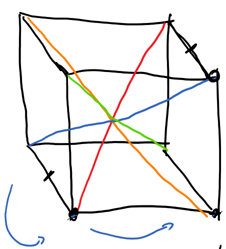
Question What are all the symmetries of this cube?

pick corner then where the edge goes

Problem How do we list these symmetries. If you do one followed by another where do you end up?

Idea Draw the long diagonals

2

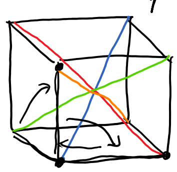


symmetry of cube ⇄ permutation of diagonals

no matter how you rotate the cube any diagonals will go to long diagonals

Converse also true!

What happens when we permute 2 diagonals? It suffices to think about this because perm product of transposition



we get that any permutation defines a symmetry. unique!

3

So there are 24 symmetries!

This is great because we know how to describe doing one permutation and then another!

Future : understand symmetries of much more complicated objects!

Idea Every group is the collection of symmetries of some geometric object — Cayley graph — Next week.

Most interesting for infinite groups!

Definition A group is a set G with a multiplication $G \times G \to G$ with prop:

1) identity $\exists\, 1 \in G$ s.t. $1 \cdot g = g \cdot 1 = g$
2) inverses $\forall x \in G\ \exists\ x^{-1} \in G$ s.t. $x \cdot x^{-1} = x^{-1} \cdot x = 1$
3) associativity $f \cdot (g\, h) = f \cdot g \cdot h = (f \cdot g) \cdot h$

What is this in terms of symmetries?

4

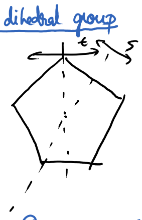
Do g then do h is the equivalent of $h \cdot g$

1 - leaves the cube alone

g^{-1} - symmetry that undoes g.

Now, let's look at some groups and see how we can think about them as symmetries.

The dihedral group



$s = s^{-1}$ $t = t^{-1}$

$(st)^m \to m$ clockwise moves

$(st)^5 = 1$

→ Coxeter groups

abstract characterization of finite groups generated by reflections groups f.d Euclidean sp.

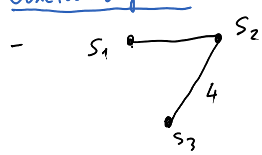
$W = \langle s_1, s_2, \ldots s_n \mid (s_i s_j)^{m_{ij}} \rangle$ $m_{ii} = 1$

$m_{ij} = \infty$ means there is no relation

5

can be represented by a matrix (m_{ij})

Coxeter diagram



$(s_1 s_2)^3 = 1$

$(s_1 s_2)^4 = 1$

$(s_1 s_3)^2 = 1$

The Symmetric Group

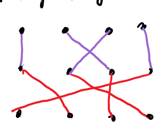
→ combinatorially permutations on $\{1, \ldots, n\}$ mult as comp.

Cycle - permutation sends $i_1 \to i_2 \to i_3 \to \ldots\ i_k \to i_1$

$(1\ 2\ 4)$ means $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}$

Every perm product of disjoint cycles.

$(1\ 2\ 4)\ (2\ 3)$



$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

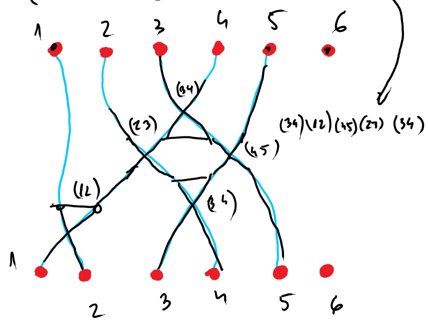
6

5

6

Transpositions - cycles length 2; any perm product of transp.

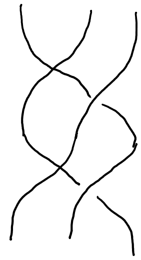
$(124)(35)$



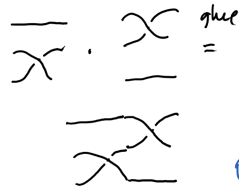
7

Braid Groups

- abstraction of things you've already seen; strings with separated ends.



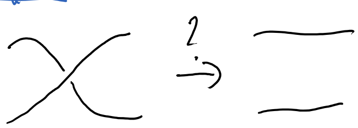
What is a product?



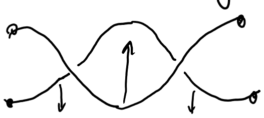
Question What is the identity

8

Harder question What is the inverse of a braid?



We need to be able to move things as in physical space. They don't go through each other and the ends stay fixed.



$\mathbb{Z}/m\mathbb{Z}$ $\{0, 1, \dots, m-1\}$ id is 0 $0 + a = a$

and the inverse of a is $m - a$.

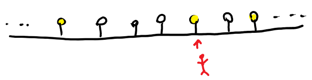
$\rightarrow m = 2$ can think of $\mathbb{Z}/2\mathbb{Z}$ as a light switch where 0 you do nothing and 1 you flip the switch.

Can see $1 + 1 = 0$.

9

Lamplighter Groups

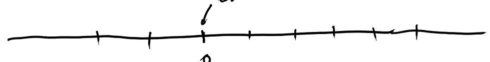
infinite street evenly spaced lampposts. Initially all are turned off. A person moves around, can change the state of finitely many and then stand at one point.



How can we describe the picture?

- one nr to say where the person is
- a tuple to say which ones are lighted

each either on or off



So we have a copy of $\mathbb{Z}_2$ at each integer and then the spot where the person is recorded in a copy of $\mathbb{Z}$

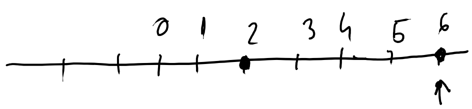
Mathematically we have $\bigoplus_{i=1}^{\infty}(\mathbb{Z}/2\mathbb{Z})$ and a copy of $\mathbb{Z}$

How can we create an element of the group:

- move left/right $\rightarrow$ t - right
- change state of lamp $\rightarrow$ a.

10

So, example $t^2 a t^4 a$



How do you multiply $g \cdot h$. Do h then g on the picture of h.

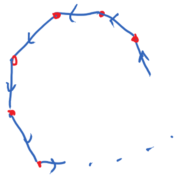
$\mathbb{Z}/m\mathbb{Z}$ - group of symmetries - n gon

Infinite groups

$\mathbb{Z}$ - symmetry of numbered line - transf preserving the distinguished points. What transf does $\mathbb{Z}$ give?

element $n \geq 0$ translate by n to right a o g on the picture of n.

$\mathbb{Z}/n\mathbb{Z}$ - group of symmetries - n gon

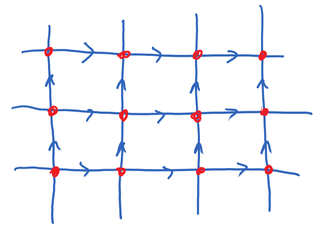


Note This shows $\mathbb{Z}/n\mathbb{Z}$ subgroup of D_n

11

element $m \geq 0$ translate by m to right

$\mathbb{Z}^2$ Same idea!



(n,m)
$\downarrow$
n to the left
m up

12

Matrix groups $GL(n,\mathbb{R})$ - matrix $n \times n$ with $\mathbb{R}$ entries and $\det \neq 0$.

Why is this a group?

$SL(n,\mathbb{R})$ — same but now $\det = 1$

Free groups

Def A word in letters a, b finite string using a, b, a^{-1}, b^{-1}. Ex: aba, $a^{-1}ba^{-1}ba$, aa^{-1}

Mult by concat.

Def A reduced word a word s.t. you never see a, a^{-1} next to each other (or b, b^{-1})

$$\underset{\text{word}}{a^{-1}aba} \longrightarrow \underset{\text{reduced word}}{ba}$$

$F_2 = \{\text{reduced words in } a, b\}$

13

Question How do you get inverse?

$$(a\,b\,a^{-1}\,b^2) \cdot (b^{-2}\,a\,b^{-1}\,a^{-1}) = id$$

Homomorphisms & Normal subgroups

Study maps between groups:

Def A homomorphism $f: G \to H$ is a map that preserves the group multiplication i.e. $f(ab) = f(a)f(b)$.

hom + bij $\longrightarrow$ isomorphism.

Note you can always restrict to $f: G \to f(G)$ to get a surjection so let's focus on injectivity part.

Examples of inj homom.:

- $\mathbb{Z}/n\mathbb{Z} \longrightarrow D_n$
  $m \longmapsto$ rotation by $2\pi/m$
- $\mathbb{Z}/2\mathbb{Z} \longrightarrow S_n$
  $1 \longmapsto$ any transposition
- $\mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$
  $n \longmapsto (na, nb)$

14

Examples of non-inj hom.

Can still be useful! We have some info given by the homomorphism mult condition.

- $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$
  $1 \longmapsto 1$
  light switch doesn't matter how many times you flipped it.
- $GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$
  $A \longmapsto \det A$
- $F_2 \longrightarrow \mathbb{Z}^2$
  $a \longmapsto (1,0)$
  $b \longmapsto (0,1)$
  remembers how many a's and b's we have.

Normal subgroups A normal subgroup of G is a group inside G s.t. $gng^{-1} \in N$ $\forall n \in N, g \in G$

Kernel of homomorphisms ——— Normal subgroups

A kernel is a normal subgroup.

$\varphi: G \longrightarrow H$ and let $N = \ker\varphi$ then if $n \in N$ is $gng^{-1} \in N$? i.e. is it in the kernel? Yes!

$$\varphi(gng^{-1}) = \varphi(g)\underbrace{\varphi(n)}_{id}\varphi(g^{-1}) = \varphi(g)\cdot\varphi(g^{-1}) = id.$$

Examples $F_2 \to \mathbb{Z}^2$ as before has kernel the elements in F_2 that have a exponents adding up to 0 (same for b).

a group inside G s.t. $gng^{-1} \in N \ \forall n \in N, g \in G$

15

16

But also, every normal subgroup is a kernel!

$G \longrightarrow G/N$ quotient group

"declare every element of N to be trivial"

$g_1 \sim g_2$ iff $g_1 g_2^{-1} \in N$

so $g \longmapsto [g]$ equivalence class.

How do we multiply in G/N? $[g_1]\cdot[g_2] = [g_1 g_2]$ for this to be well defined we need the prop of normal subgroup.

First isom thm $G \xrightarrow{\varphi} H$ surj hom then

$G/\ker\varphi \cong H$.

17

Group presentations

Instead of writing the whole mult notice some entries imply others.

$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n = 1 \rangle$ → every entry follows from this

$\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$

Presentation - pair (S, R) elements are in free group $F(S)$ with relations R.

$G \cong \langle \underset{\text{generators}}{S} \mid \underset{\text{relators}}{R} \rangle$ iff $F(S)/_{\text{normal closure of } R} \cong G$

So for example if we have relation $ab = ba$ then we get the relator $aba^{-1}b^{-1}$

$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$

Next talk: Cayley graphs, showing that the group of symmetries of this Cayley graph is indeed the orig group
by Zhenfeng

18